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On the homotopy classification of sections in the free loop fibration

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For Stephen Halperin on his 50th birthday

Abstract

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For a class of spaces including simply connected rational spaces the homotopy classification of sections in the corresponding free loop fibration is given.

1. Introduction

In the free loop fibration over a topological space X, $\Omega X \to \Lambda X \xrightarrow{\zeta} X$, where ΛX denotes the space of all maps of the circle S^1 into X (not based), and ΩX the based loop space, there is a canonical section $s: X \to \Lambda X$ assigning to a point of X the constant loop into it. So a natural problem is to classify all sections up to homotopy. This problem is, in general, difficult, but in some cases we consider, there is an algorithm for that. Namely, the classification is based on the examination with the Hirsch model (complex) of the fibration ζ [1, 3, 4, 7]

$$k:(C_*(X)\otimes H_*(\Omega X), \partial_h) \to C_*(\Lambda X),$$

where ΩX has the homotopy type of a product of Eilenberg–Mac Lane spaces, $K(\pi_n, n)$'s, π_n is a vector space over a fixed coefficient field $\mathbf{k}$. Moreover, a perturbed differential ∂_h is determined via a twisting cochain [1, 3]

$$h \in (C^{*+1}(X; \mathrm{Hom}^*(H_*(\Omega X), H_*(\Omega X))), d)$$

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by the formula $\partial_h = \partial + h \cap \_$. However, since ζ has a section, we can choose h such that it will be a cochain with coefficients in those homomorphisms which preserve the direct summand of the homotopy groups (Proposition 2.2). So that we obtain an induced *spherical twisting cochain*:

$$v \in (C^{*+1}(X; \mathrm{Hom}^*(\pi_*(\Omega X), \pi_*(\Omega X))), d)$$

which defines a new differential, d_v, in the cochain complex $(C^{*+i}(X; \pi_*(\Omega X)), d)$, $i \in \mathbb{Z}$, by the formula $d_v = d + v \cup \_$. The obtained complex will be denoted by $(L^i(\zeta), d_v)$.

We have the following theorem:

Theorem 1.1. *By the hypothesis and notations above let a simply connected space X having the homotopy type of a polyhedron satisfy the following two conditions:*

(i) *ΩX has the homotopy type of a product of Eilenberg–Mac Lane spaces. $\Omega X \simeq \prod_n K(\pi_n, n)$, π_n is a vector space over* $\mathbf{k}$.

(ii) *The free loop fibration over X has the Hirsch model with*

$$\partial_h(C_*(X) \otimes \bar{H}_*(\Omega X)) \subset C_*(X) \otimes \bar{H}_*(\Omega X),$$

where $H_*(\Omega X) = \pi_*(\Omega X) \oplus \bar{H}_*(\Omega X)$.

Then there is a bijection

$$[X, \Lambda X]_s \approx H^0(L^*(\zeta), d_v),$$

where $[-,-]_s$ denotes the set of the homotopy classes of sections.

If a space X is $\mathbf{k}$-*formal* (cf. [5], for example), then we have the following:

Theorem 1.2. *By the hypothesis of Theorem* 1.1, *if a space X is* $\mathbf{k}$-*formal, then the spherical twisting cochain v defines a differential, $\bar{v}$, on the graded vector space*

$$H^{*+i}(X; \pi_*(\Omega X)), \quad i \in \mathbb{Z},$$

such that there is a bijection

$$[X, \Lambda X]_s \approx H^0(H^{*+i}(X; \pi_*(\Omega X)), \bar{v}).$$

These theorems are special cases of Theorems 2.3 and 2.4 respectively stated for Serre fibrations in Section 2 and which themselves are particular cases of an obstruction theory for the section problem with integral coefficients [10]. However, the motivation of this note is that the free loop fibration over a simply connected rational space satisfies the hypothesis of Theorem 2.3 (Proposition 3.3), and other examples outside the rational homotopy theory are discussed in Section 3, too.

2. The homotopy classification of sections in a Serre fibration with a fibre of type a product of $K(\pi_n, n)$'s

We fix a ground field $\mathbf{k}$ and for a polyhedron (space) X, $C(X)$ will denote the simplicial (singular) complex with coefficients in $\mathbf{k}$.

Let

$$F \to E \xrightarrow{\xi} X$$

be a Serre fibration, where X is a polyhedron, F has the homotopy type of a product of Eilenberg–Mac Lane spaces, $K(\pi_n, n)$'s, π_n is a vector space over $\mathbf{k}$ and $\pi_1(X)$ acts trivially on $H(F)$.

We recall the Hirsch model (complex) of the fibration ξ [1, 3, 4, 7],

$$k:(C_*(X; H_*(F)), \partial_h) \to C_*(E),$$

in which k is a homology isomorphism and a twisted (perturbed) differential is defined via a twisting cochain (of total degree 1)

$$h \in \prod_{i>0} C^{i+1}(X; \mathrm{Hom}^i(H_*(F), H_*(F)))$$

by the formula $\partial_h = \partial_C + h \cap \_$. (Using the cap-product and the evaluation map in coefficients.) So, the h has the form $h = \{h^2, h^3, \ldots, h^r, \ldots\}$ where r is called as the *perturbation degree* of h and then the twisting cochain condition, $d(h) = -hh$, implies a sequence of equalities,

$$d(h^2) = 0, \quad d(h^3) = -h^2h^2, \quad d(h^4) = -h^2h^3 - h^3h^2, \quad \ldots,$$

where the product is defined by the cup-product and composition of homomorphisms in coefficients.

Of course, a twisting cochain is not uniquely defined by ξ, but there is the following proposition:

Proposition 2.1 [1]. *For any two Hirsch models of a fibration ξ, there exists an isomorphism of complexes*

$$p:(C_*(X; H_*(F)), \partial_{\bar{h}}) \to (C_*(X; H_*(F)), \partial_h)$$

of the form $p = 1 + p'$, where p' increases the $H_*(F)$-degree, and is defined by some element from $C^*(X; \mathrm{Hom}^*(H_*(F), H_*(F)))$ (denoted also by the same symbol p) via the $\cap$ _ product. $\square$

In other words, the twisting cochains h and $\bar{h}$ are on the same orbit with respect to the action of the group of such automorphisms, p, on the set of all twisting cochains, h, via $p * h = php^{-1} + d(p)p^{-1}$. The obtained quotient set is denoted by $D(X; H_*(F))$ and in this set to the fibration ξ the element, $d(\xi)$, called the (homological) pre-differential of ξ, is still assigned [1].

Here our first observation is the following:

Proposition 2.2. *By the hypothesis above, if the fibration ξ has a section, then there is a representative (twisting cochain), $h \in d(\xi)$, such that in the Hirsch model*

$$\partial_h(C_*(X; \pi_*(F))) \subset C_*(X; \pi_*(F)).$$

From this follows that the restriction of the h to the

$$(C^{*+1}(X; \mathrm{Hom}^*(\pi_*(F), \pi_*(F))), d)$$

defines a twisting cochain, v, called as *spherical*.

Now we can form the following complex,

$$\cdots \to L^i(\xi) \xrightarrow{d_v} L^{i+1}(\xi) \to \cdots$$

in which $L^i(\xi) = \prod_n C^{n+i}(X; \pi_n(F))$, $d_v = d + v \cup \_$.

We have the following theorem:

Theorem 2.3. *By the hypothesis and notations above let a Serre fibration ξ with a section satisfy:*

(i) *F has the homotopy type of a product of Eilenberg–Mac Lane spaces, $K(\pi_n, n)$'s, π_n is* **k**-*module.*

(ii) *There is a twisting cochain $h \in d(\xi)$ such that in the Hirsch model of ξ*

$$\partial_h(C_*(X; \bar{H}_*(F))) \subset C_*(X; \bar{H}_*(F))$$

where $H_*(F) = \pi_*(F) \oplus \bar{H}_*(F)$.

Then there is a bijection

$$[X, E]_s \approx H^0(L^*(\xi), d_v)$$

where v is some spherical twisting cochain, $[-,-]_s$ denotes the set of homotopy classes of sections.

As it was mentioned above this theorem is a special case of that in [10], but for convenience we prove it here. We begin with proving Proposition 2.2.

Proof of Proposition 2.2. Recall the method of construction of a twisting cochain for a Serre fibration ξ [1] (cf. [8]). We have that ξ defines a colocal system of singular chain complexes on the base X: To each simplex $\sigma \in X$ is assigned the complex $(C_*(F_\sigma), \gamma_\sigma)$, $F_\sigma = \xi^{-1}(\sigma)$ and to a pair $\tau \subset \sigma$ the induced homomorphism $C_*(F_\tau) \to C_*(F_\sigma)$. Then $\sigma \to \mathrm{Hom}(H_*(F), C_*(F_\sigma))$ also forms a colocal system on X. Define, $\mathscr{K}$, canonically as the simplicial cochain complex of X with coefficients in the last colocal system:

$$\mathscr{K} = \{\mathscr{K}^{i,j}\}, \quad \mathscr{K}^{i,j} = C^i(X; \mathrm{Hom}^j(H_*(F), C_*(F_\sigma))).$$

Hence, $\mathscr{K}$ is a bicomplex with differentials

$$\delta : \mathscr{K}^{i,j} \to \mathscr{K}^{i+1,j}, \qquad \gamma : \mathscr{K}^{i,j} \to \mathscr{K}^{i,j-1}.$$

Denoting $(\mathscr{H}^{*,*}, d) = (C^*(X; \mathrm{Hom}^*(H_*(F), H_*(F))), d)$ we have a natural d.g. pairing

$$(\mathscr{K}, \delta + \gamma) \otimes (\mathscr{H}, d) \to (\mathscr{K}, \delta + \gamma).$$

Consider the following equation,

$$(\delta + \gamma)(k) = kh,$$

with the initial conditions

$$d(h) = -hh,$$

$$h = \{h^2, \dots, h^r, \dots\}, \quad h^r \in \mathscr{H}^{r,r-1},$$

$$k = \{k^0, \dots, k^r, \dots\}, \quad k^r \in \mathscr{K}^{r,r},$$

$$\gamma(k^0) = 0, \qquad [k^0]_\gamma = 1 \in \mathscr{H}^{0,0}.$$

Here in addition we require that the components of k,

$$k_0^r \in C^r(X; \mathrm{Hom}(H_0(F), C_r(F_\sigma))) = C^r(X; C_r(F_\sigma)),$$

are determined by a given section, s, that means

$$k_0^r(\sigma^r) = s|_{\sigma^r}, \quad \sigma^r \in X.$$

Moreover, we suppose for each generator $a \in \pi_i(F)$, $k_i^0(\sigma^0)(a) \in C_i(F_{\sigma^0})$ is a spheroid, $(S^i, x_0) \to (F_{\sigma^0}, k_0^0(\sigma^0))$, representing a.

Now we begin to find a solution, a pair (k, h), of the equation where h will preserve the homotopy groups in coefficients. This process goes by induction on the perturbation degree. First consider $\delta(k^0) \in \mathscr{K}^{1,0}$. By hypothesis we can find some $k^1 \in \mathscr{K}^{1,1}$ with $\gamma(k) = -\delta(k^0)$, while for a spherical generator we take $k_i^1(\sigma^1)(a) \in C_{i+1}(F_{\sigma^1})$ to be a homotopy, $G: S^i \times I \to F_{\sigma^1}$, such that $G|_{x_0 \times I} = k_0^1(\sigma^1)$ $(= s|_{\sigma^1})$. Next consider $\delta(k^1)$. Then we have $\gamma\delta(k^1) = -\delta\gamma(k^1) = \delta\delta(k^0) = 0$. So, we obtain the cochain,

$$h^2 = [\delta(k^1)]_\gamma, \quad h^2 = \{h_i^2\} \in \mathscr{H}^{2,1}, \quad h_i^2 \in C^2(X; \mathrm{Hom}(H_i(F), H_{i+1}(F))),$$

with the property that for $a \in \pi_i(F)$, $h_i^2(\sigma^2)(a) \in \pi_{i+1}(F)$. In fact, the chain $\delta(k_i^1)(\sigma^2)(a)$ factors through a map,

$$\alpha: T_{\sigma^2}^{i+1}(a) \to F_{\sigma^2}, \qquad T_{\sigma^2}^{i+1}(a) = \partial\Delta^2 \times S^i \cup_{\partial\Delta^2 \times x_0} \Delta^2,$$

and there is a spheroid, $\beta: S^{i+1} \to T_{\sigma^2}^{i+1}(a)$, representing the generator of $H_{i+1}(T_{\sigma^2}^{i+1}(a))$.

Next we consider $k^0h^2 - \delta(k^1)$. Clearly, there is some $k^2 \in \mathscr{K}^{2,2}$ with $\gamma(k^2) = k^0h^2 - \delta(k^1)$, while for a generator $a \in \pi_i(F)$, we choose $k_i^2(\sigma^2)(a) = Z_1 + Z_2$ where Z_1 is a chain with $\gamma_{\sigma^2}(Z_1) = \alpha \circ \beta - \alpha$ factoring through α and Z_2 is a homotopy between $\alpha \circ \beta$ and $k^0h^2(\sigma^2)(a)$.

Continuing this process we will have that $h^n \in \mathscr{H}^{n,n-1}$ is defined as the homology class of the γ-cocycle,

$$f^n = \delta(k^{n-1}) - k^{n-2}h^2 - \cdots - k^1h^{n-1},$$

while $k^n \in \mathscr{K}^{n,n}$ is defined by $\gamma(k^n) = k^0h^n - f^n$; moreover, for a generator $a \in \pi_i(F)$, $h_i^n(\sigma^n)(a) \in \pi_{i+n-1}(F)$, since one can immediately see that $f_i^n(\sigma^n)(a) \in C_{i+n-1}(F_{\sigma^n})$ factors through a map, $\alpha: T_{\sigma^n}^{i+n-1}(a) \to F_{\sigma^n}$, where the generator of $H_{i+n-1}(T_{\sigma^n}^{i+n-1}(a))$ is defined by a spheroid, $\beta: S^{i+n-1} \to T_{\sigma^n}^{i+n-1}(a)$ (in fact, $T_{\sigma^n}^{i+n-1}(a)$ has the homotopy type of a bouquet of spheres with only one component of S^{i+n-1}). Also we define $k_i^n(\sigma^n)(a) = Z_1 + Z_2$ entirely analogously to k_i^2. Thus, we obtain a solution (k, h) of the equation with h having the property that $h(\sigma)(a) \in \pi_*(F)$, $a \in \pi_*(F)$, $\sigma \in X$. The proposition is proved. □

Proof of Theorem 2.3. First remark that we can choose $h \in d(\xi)$ such that in the Hirsch model ∂_h satisfies conclusions of Proposition 2.2 and part (ii) of the theorem at the same time. Then define a map,

$$\psi : [X, E]_s \to H^0(L^*(\xi), d_v),$$

as follows. Suppose some section, s, of ξ is fixed. Let s' be any section of ξ. Consider a fibration, ξ', over $X \times I$ induced from ξ by the projection $X \times I \to X$. Then we consider the equation above for ξ' with initial conditions where in addition we fix a given solution (k, h) on $X \times 0$ and $k_0^*(\sigma) = s'|_\sigma$ on $X \times 1$. Then we obtain a twisting cochain, h', determined by the equation. (In general, one can see from the proof of Proposition 2.2 that any given solution on a subcomplex extends to the whole complex.) Now define a sequence of cochains,

$$\{c^i\}_{s'}, \quad c^i \in C^i(X; \pi_i(F)),$$

where $c^i(\sigma^i)$ is the restriction of $h_0'^{\,i+1}(\sigma^i \times I) \in H_i(F)$ to $\pi_i(F)$ (here we use the standard cellular decomposition of the cylinder). Consequently, we will have that the twisting cochain condition implies in $L(\xi)$, $d_v\{c^i\} = 0$. Moreover, if the section s' is homotopic to other section, s'', then we consider a fibration, ξ'', over $X \times I \times I$ induced from ξ by the projection $X \times I \times I \to X$, and fix this homotopy on $X \times I \times 1$ (realizing as k_0^*), while a solution for ξ' on $X \times I \times 0$. Let h'' be an obtained twisting cochain for ξ''. Then for cochains $\theta^i \in C^i(X; \pi_i(F))$, $\theta^i(\sigma^i)$ is the restriction of $h_0''^{\,i+2}(\sigma^i \times I \times I)$ to $\pi_{i+1}(F)$, we will have $d_v\{\theta^i\} = \{c^i\}_{s'} - \{c^i\}_{s''}$. Thus, the assignment $s' \to \{c^i\}_{s'}$ induces the map ψ above.

Conversely, we assign to a d_v-cocycle $\{c^i\} \in L^i(\xi)$ a section of ξ as follows. Consider again the equation for ξ' over $X \times I$, where on $X \times 0$ the section s fixes k_0^*, and begin to construct a solution (k, h). Choose k^1 such that k_0^1 is realized on $X^1 \times 1$ by some section $X^1 \to E$ of ξ with $\delta(k_0^1)(\sigma^1 \times I)$ a representative of $c^1(\sigma^1)$, or, $h_0^2(\sigma^1 \times I) = c^1(\sigma^1)$. Suppose we have defined k^i, $i = 1, \ldots, n-1$, and h^i, $i = 1, \ldots, n$, with $k_0^{n-1}|_{X^{n-1} \times 1}$ realized by some section of X^{n-1} and

$$h_0^i|_{X^n \times 1} = 0, \qquad h_0^i(\sigma^{i-1} \times I)|_{\pi_{i-1}(F)} = c^{i-1}(\sigma^{i-1}).$$

Then an extension of the section k_0^{n-1} exists on $X^n \times 1$ (since $h_0^n = [\delta(k_0^{n-1})]_\gamma$ is just the obstruction cocycle, cf. [2]) and choose $k_0^n(\sigma^n \times 1)$ with the property that the restriction of $h_0^{n+1}(\sigma^n \times I)$ to $\pi_n(F)$ is $c^n(\sigma^n)$. Moreover, the twisting cochain condition shows that $h_0^{n+1}|_{X^{n+1} \times 1}0$. Consequently, we have constructed a section, s', on $X \times 1$ which is clearly defined up to homotopy. Now if the $\{c^i\}$ were a d_v-boundary, then by considering the fibration ξ'' over $X \times I \times I$ as above we would obtain that s' is homotopic to the s. Therefore, a map, $H^0(L^*(\xi), d_v) \to [X, E]_s$, is defined, which is obviously the converse of ψ. The theorem is proved. □

Computations with this theorem simplifies if X is a $\mathbf{k}$-formal space, that is, there exists an algebra map,

$$\chi:(R^*H^*(X), d) \to C^*(X;\mathbf{k})$$

inducing an isomorphism in cohomology where

$$\rho:(R^*H^*(X), d) \to H^*(X)$$

is a multiplicative (non-commutative) resolution (for example, $(\bar{\Omega}\bar{B}H(X), d) \to H(X)$, the Bar–Cobar resolution of the graded algebra $H(X)$ (cf. [5, 9]). In this case, the algebra maps χ and ρ transfer a (spherical) twisting cochain into a (spherical) twisting element in $H^{*+1}(X; \mathrm{Hom}^*(H_*(F), H_*(F)))$ using the argument similar to that of the proof of the comparison theorem for the functor D [1] (see also [8, Theorem 4.1]). Let $\bar{v}$ be a so obtained twisting element from the spherical twisting cochain v in Theorem 2.3. Then $\bar{v}$ defines the complex by the natural pairing (where a differential is denoted by the same symbol):

$$\cdots \to H^{*+i}(X;\pi_*(F)) \xrightarrow{\bar{v}} H^{*+i+1}(X;\pi_*(F)) \to \cdots$$

and we will have a natural isomorphism

$$H^*(L^i(\xi), d_v) \approx H^*(H^{*+i}(X;\pi_*(F)), \bar{v}).$$

Consequently, from Theorem 2.3 follows the following:

Theorem 2.4. *If a fibration ξ is as in Theorem 2.3 and the base X is $\mathbf{k}$-formal, then there is a bijection*

$$[X, E]_s \approx H^0(H^{*+i}(X;\pi_*(F)), \bar{v}). \qquad \square$$

3. Applications and examples

As an immediate consequence of Theorem 1.2 we have the following proposition:

Proposition 3.1. *Let a 2-connected space X be as in Theorem 1.2 and $H^*(X)$ has the trivial multiplication. Then there is a bijection*

$$[X, \Lambda X]_s \approx \prod_n H^n(X;\pi_{n+1}(X)). \qquad \square$$

However, there is the following:

Example 3.2. Let X be the homotopy fibre in

$$X \to K(\mathbb{Z}_p, n) \xrightarrow{f} K(\mathbb{Z}_p, 2n), \quad n > 1,$$

where $f^*(e_{2n}) = e_n^2$, e_n is the fundamental class. Then it is easy to verify that X satisfies the hypothesis of Theorem 1.1 and $[X, \Lambda X]_s \approx H^{2n-2}(X; \mathbb{Z}_p)$.

More generally, a space X satisfies the hypothesis of Theorem 1.1 if it has a Postnikov tower (X_n, ξ_n) with k-invariants, $k_n : X_n \to K(\pi_{n+1}, n+2)$:

(i) $(\Omega k_n)^*(e_{n+1}) = 0 \in H^{n+1}(\Omega X_n; \pi_{n+1}(X))$,

(ii) the composition map, k_n', formed by the evaluation map $\Lambda X_n \times S^1 \to X_n$ and k_n satisfies

$$k_n'^*(e_{n+2}) = u \otimes t \in H^{n+1}(\Lambda X_n) \otimes H^1(S^1)$$

where u has a representative in the subspace $C^*(X) \otimes \pi^*(\Omega X)$ of the Hirsch complex of ΛX_n, $\pi^* = \mathrm{Hom}(\pi_*, \mathbf{k})$.

In the rational case, we have the following:

Proposition 3.3. *If X is a simply connected rational space having the homotopy type of a polyhedron with the finite-dimensional homotopy groups in each degree, then it satisfies the hypothesis of Theorem* 1.1 *and, consequently, there is a bijection*

$$[X, \Lambda X]_s \approx H^0(L^*(\zeta), d_v).$$

Proof. Recall [11] that if $(\mathscr{A}(X), d)$ is the Sullivan minimal model of X, then the minimal model of ΛX is $(\mathscr{A}(X) \otimes H(\Omega X), d_\tau)$, $d_\tau = d + \tau$, $\tau(\bar{z}) = -i(d(z))$, for a generator $\bar{z} \in H^*(\Omega X)$, where $i : \mathscr{A}(X) \to \mathscr{A}(X) \otimes H(\Omega X)$ is the derivation of degree -1 extending the bijection $z \to \bar{z}$. There are maps of algebras

$$\mathscr{A}^*(X) \leftarrow (R^*(H^*(X), d') \to C^*(X; \mathbb{Q})$$

inducing isomorphisms in cohomology, where $(R^*(H^*(X), d')$ is the (non-commutative) filtered model of X (compare [6, 9]). These maps transfer (cf. the proof of Theorem 2.4) the perturbation

$$\tau \in \mathscr{A}^{*+1}(X) \hat{\otimes} \mathrm{Hom}^*(H^*(\Omega X), H^*(\Omega X))$$

into a twisting cochain (cohomological) in

$$C^{*+1}(X; \mathrm{Hom}^*(H^*(\Omega X), H^*(\Omega X))),$$

and, consequently, the dual (homological) twisting cochain will be as desired. The proposition is proved. □

Remark 3.4. The bijection in this proposition can be established only by means of models in the rational homotopy theory. In particular, one can verify that $H^*(L^*(\zeta), d_v)$ is naturally isomorphic to the $H^*(\tilde{L}(h), \tilde{V})$ in [9, Section 4].

Example 3.5. Let X be the rationalization of $(S^2 \vee S^2) \times S^2$ and let us compute $[X, \Lambda X]_s$. For the minimal model of X, we have

$$(\mathscr{A}, d) = \Lambda(x, y, w, t_1, t_2, t_3, v_1, v_2, u_1, u_2, u_3, \ldots, d),$$

$x, y, w \in \mathscr{A}^2$, $t_1, t_2, t_3 \in \mathscr{A}^3$, $v_1, v_2 \in \mathscr{A}^4$, $u_1, u_2, u_3 \in \mathscr{A}^5$, with

$$\begin{aligned}
&0 = dx = dy = dw, && dv_1 = t_2 x - t_1 y,\\
&dt_1 = x^2, && dv_2 = t_2 y - t_3 x,\\
&dt_2 = xy, && du_1 = v_1 x + t_1 t_2,\\
&dt_3 = y^2, && du_2 = v_1 y - v_2 x + t_3 t_1,\\
& && du_u = v_1 y - t_3 t_2.
\end{aligned}$$

Let $(\mathscr{A}(X) \otimes H(\Omega X), d + \tau)$ be the minimal model of ΛX. Then the perturbation τ induces by an algebra map $(\mathscr{A}(X), d) \to H(X)$ (since X is formal) a twisting element, $\bar{v}$, in $H^{*+1}(X) \otimes \mathrm{Hom}^*(\pi_*(\Omega X), \pi_*(\Omega X))$ of the form (we denote the dual of the generator of $H^*(\Omega X)$ by the same symbol):

$$\begin{aligned}
\bar{v} = \bar{v}^2, \qquad & \bar{v}^2 \in H^2(X) \otimes \mathrm{Hom}^1(\pi_*(\Omega X), \pi_*(\Omega X)),\\
& \bar{v}^2 = x \otimes \phi_x + y \otimes \phi_y,
\end{aligned}$$

$$\begin{aligned}
&\phi_x(\bar{x}) = -2\bar{t}_1, && \phi_x(\bar{t}_2) = -\bar{v}_1, && \phi_x(\bar{v}_1) = -\bar{u}_1,\\
&\phi_x(\bar{y}) = -\bar{t}_2, && \phi_x(\bar{t}_3) = \bar{v}_2, && \phi_x(\bar{v}_2) = \bar{u}_2,\\
&\phi_y(\bar{x}) = -\bar{t}_2, && -\phi_y(\bar{t}_1) = \bar{v}_1, && \phi_y(\bar{v}_1) = -\bar{u}_2,\\
&\phi_y(\bar{y}) = -2\bar{t}_3, && \phi_y(\bar{t}_2) = -\bar{v}_2, && \phi_y(\bar{v}_2) = -\bar{u}_3.
\end{aligned}$$

Then in the complex $(H^{*+i}(X) \otimes \pi_*(\Omega X), \bar{v})$ we have that all 0-dimensional elements are cocycles except those generated by $w \otimes \bar{t}_j, j = 1, 2, 3$. Moreover,

$$\bar{v}(1 \otimes \bar{x}) = -2x \otimes \bar{t}_1 - y \otimes \bar{t}_2,$$

$$\bar{v}(1 \otimes \bar{y}) = -x \otimes \bar{t}_2 - 2y \otimes \bar{t}_3,$$

$$\bar{v}(w \otimes \bar{v}_1) = -xw \otimes \bar{u}_1 - yw \otimes \bar{u}_2,$$

$$\bar{v}(w \otimes \bar{v}_2) = xw \otimes \bar{u}_2 - yw \otimes \bar{u}_3.$$

Thus, $[X, \Lambda X]_s \approx H^0(H^{*+i}(X) \otimes \pi_*(\Omega X), \bar{v})$ is the 8-dimensional rational vector space with a basis

$$\{x \otimes \bar{t}_i, y \otimes \bar{t}_1, xw \otimes \bar{u}_i, yw \otimes \bar{u}_1\}, \quad i = 1, 2, 3.$$

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